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On the Laplacian spectral ratio of connected graphs

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ABSTRACT

The Laplacian spectrum of a graph is the eigenvalues of the associated Laplacian matrix. The quotient between the largest and second smallest Laplacian eigenvalues of a connected graph, is called the Laplacian spectral ratio. Some bounds on the Laplacian spectral ratio are considered. We improve a relation on the Laplacian spectral ratio of regular graphs. Especially, the first two smallest Laplacian spectral ratios of graphs with given order are determined. And some operations on Laplacian spectral ratio are presented.

1. Introduction

Let G be a simple graph with n vertices and m edges. Denote by $\delta(G)$ and $\Delta(G)$ the minimum and the maximum degree, respectively. The Laplacian matrix of G is defined as $L = D - A$, where $D = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ is the diagonal matrix of vertex degrees and A is the adjacency matrix of G. The Laplacian spectrum of G is the spectrum of its Laplacian matrix, and consists of the values $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$. Since $\mu_{n-1}(G) > 0$ if and only if G is connected, Fiedler [1] called $\mu_{n-1}(G)$ (or $\alpha(G)$) the algebraic connectivity of G.

The Laplacian spectral ratio of a connected graph G with n vertices is defined as

$$r_L(G) = \frac{\mu_1}{\mu_{n-1}}.$$

In 2002, Barahona et al. [2] showed that a graph exhibits better synchronizability if the ratio $r_L(G) = \frac{\mu_1}{\mu_{n-1}}$ is as small as possible.

Theorem A ([3]). *If $G \neq K_n$ is a connected graph with n vertices, then*

$$r_L(G) = \frac{\mu_1(G)}{\mu_{n-1}(G)} \geq \frac{\Delta(G)+1}{\delta(G)}.$$

A vertex cut of G is a subset S of $V(G)$ such that $G - S$ is disconnected. G is a t-tough graph ($t > 0$ and $t \in R$) if, for every vertex cut S, the number of components of the graph $G - S$, denoted by $C(G - S)$, is at most $|S|/t$, i.e., $C(G - S) \leq |S|/t$.

Theorem B ([4]). *Let G be a simple graph with n vertices, and Laplacian eigenvalues $0 = \mu_n \leq \mu_{n-1} \leq \cdots \leq \mu_1$. If $\mu_{n-1} \geq \frac{2}{3}\mu_1$, then G is 2-tough.*

From Theorem B, we can see that if $r_L(G) \leq \frac{3}{2}$, then G is 2-tough.

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In this paper we obtain some bounds on the Laplacian spectral ratio. A relation on the Laplacian spectral ratio of regular graphs is improved. A problem on the extremal Laplacian spectral ratio among trees with n vertices is proposed. Moreover, some graph operations on the Laplacian spectral ratio are given.

Lemma 1.1 ([5]). *Let X and Y be disjoint sets of vertices of G, such that there is no edge between X and Y. Then* $\frac{|X|\,|Y|}{(n-|X|)(n-|Y|)} \le (\frac{\mu_1-\mu_{n-1}}{\mu_1+\mu_{n-1}})^2$.

Proposition 1.1. *Let X and Y be disjoint sets of vertices of a connected graph G, such that there is no edge between X and Y. Then* $r_L(G) \ge \frac{2\sqrt{(n-|X|)(n-|Y|)}}{\sqrt{(n-|X|)(n-|Y|)}-\sqrt{|X|\,|Y|}} - 1$.

Proof. By Lemma 1.1, $\frac{r_L-1}{r_L+1} = \frac{\mu_1-\mu_{n-1}}{\mu_1+\mu_{n-1}} \ge \sqrt{\frac{|X|\,|Y|}{(n-|X|)(n-|Y|)}}$.

Then $1 - \frac{2}{r_L+1} \ge \sqrt{\frac{|X|\,|Y|}{(n-|X|)(n-|Y|)}}$.

By the above inequality, the result follows. □

Lemma 1.2 ([5]). *If G is a connected graph with diameter $d > 1$, then*

$$d < 1 + \frac{\log 2(n-1)}{\log(\sqrt{\mu_1}+\sqrt{\mu_{n-1}}) - \log(\sqrt{\mu_1}-\sqrt{\mu_{n-1}})}.$$

Proposition 1.2. *Let G be a connected graph with n vertices and diameter $d > 1$. Then*

$$r_L > \left(1 + \frac{2}{\sqrt[d-1]{2(n-1)} - 1}\right)^2.$$

Proof. By Lemma 1.2, $n - 1 > \frac{1}{2}(\frac{\sqrt{\mu_1}-\sqrt{\mu_{n-1}}}{\sqrt{\mu_1}+\sqrt{\mu_{n-1}}})^{d-1}$.

Then $\sqrt[d-1]{2(n-1)} > \frac{\sqrt{r_L}+1}{\sqrt{r_L}-1} = 1 + \frac{2}{\sqrt{r_L}-1}$.

The above inequality transforms into the result by solving for r_L. □

2. Bounds on the Laplacian spectral ratio

Note that $\sum_{i=1}^n \mu_i^2 = \sum_{u\in V} d_u^2 + 2m := M_1 + 2m$.

Lemma 2.1 ([6] Unweighted Cassels' Inequality). *Let $\overline{a} = (a_1, \dots, a_n)$ and $\overline{b} = (b_1, \dots, b_n)$ be two positive n-tuples. Then*

$$\frac{\left(\sum_{i=1}^n a_i^2\right)\left(\sum_{i=1}^n b_i^2\right)}{\left(\sum_{i=1}^n a_i b_i\right)^2} \le \frac{(a+A)^2}{4aA},$$

where $a = \min_{\overline{1,n}}\{\frac{a_i}{b_i}\}$ and $A = \max_{\overline{1,n}}\{\frac{a_i}{b_i}\}$.

Theorem 2.1. *Let G be a connected graph with n vertices and m edges. Then*

$$\sqrt{r_L} + \sqrt{\frac{1}{r_L}} \ge \frac{\sqrt{(n-1)(M_1+2m)}}{m}.$$

Proof. Let $\overline{a} = (1, \dots, 1)$ and $\overline{b} = (\mu_{n-1}, \dots, \mu_1)$ be two positive $n-1$-tuples. Then $a = \frac{1}{\mu_1}$ and $A = \frac{1}{\mu_{n-1}}$.

By Lemma 2.1,

$$\sqrt{\frac{(n-1)\sum_{i=1}^{n-1}\mu_i^2}{\left(\sum_{i=1}^{n-1}\mu_i\right)^2}} \le \frac{a+A}{2\sqrt{aA}}, \quad \text{i.e.,}$$

$$\sqrt{\frac{(n-1)(M_1+2m)}{(2m)^2}} \le \frac{a+A}{2\sqrt{aA}} = \frac{1}{2}\left(\sqrt{\frac{A}{a}} + \sqrt{\frac{a}{A}}\right) = \frac{1}{2}\left(\sqrt{r_L} + \sqrt{\frac{1}{r_L}}\right).$$

The result follows. □

Corollary 2.1 ([3]). *Let G be a connected k-regular graph with n vertices. Then*

$$\sqrt{r_L}+\sqrt{\frac{1}{r_L}}\geq 2\sqrt{\frac{n-1}{n}\frac{k+1}{k}}.$$

Remark 2.1. By Theorem A, Goldberg [3] obtained that for a k-regular graph G,

$$\sqrt{\frac{\mu_1(G)}{\mu_{n-1}(G)}}+\left(\sqrt{\frac{\mu_1(G)}{\mu_{n-1}(G)}}\right)^{-1}\geq\sqrt{\frac{k+1}{k}}+\sqrt{\frac{k}{k+1}}. \tag{1}$$

For $n\geq 3$, it is easy to check that $2\sqrt{\frac{n-1}{n}\frac{k+1}{k}}>\sqrt{\frac{k+1}{k}}+\sqrt{\frac{k}{k+1}}$. Then the bound of Corollary 2.1 is better than (1).

The following additive version of unweighted Cassels' inequality also holds.

Lemma 2.2 ([6]). *Let $\overline{a}=(a_1,\dots,a_n)$ and $\overline{b}=(b_1,\dots,b_n)$ be two positive n-tuples. Then*

$$\left(\sum_{i=1}^{n}a_i^2\right)\left(\sum_{i=1}^{n}b_i^2\right)-\left(\sum_{i=1}^{n}a_ib_i\right)^2\leq\frac{(A-a)^2}{4aA}\left(\sum_{i=1}^{n}a_ib_i\right)^2,$$

where $a=\min_{\overline{1,n}}\{\frac{a_i}{b_i}\}$ and $A=\max_{\overline{1,n}}\{\frac{a_i}{b_i}\}$.

Theorem 2.2. *Let G be a connected graph with n vertices and m edges. Then*

$$\sqrt{r_L}-\sqrt{\frac{1}{r_L}}\geq\frac{\sqrt{(n-1)(M_1+2m)-4m^2}}{m}.$$

Proof. Let $\overline{a}=(1,\dots,1)$ and $\overline{b}=(\mu_{n-1},\dots,\mu_1)$ be two positive $n-1$-tuples, $a=\frac{1}{\mu_1}$ and $A=\frac{1}{\mu_{n-1}}$.

By Lemma 2.2,

$$\sqrt{\frac{(n-1)\sum\limits_{i=1}^{n-1}\mu_i^2-\left(\sum\limits_{i=1}^{n-1}\mu_i\right)^2}{\left(\sum\limits_{i=1}^{n-1}\mu_i\right)^2}}\leq\frac{A-a}{2\sqrt{aA}},\quad\text{i.e.,}$$

$$\sqrt{\frac{(n-1)(M_1+2m)-4m^2}{(2m)^2}}\leq\frac{A-a}{2\sqrt{aA}}=\frac{1}{2}\left(\sqrt{\frac{A}{a}}-\sqrt{\frac{a}{A}}\right)=\frac{1}{2}\left(\sqrt{r_L}-\sqrt{\frac{1}{r_L}}\right).$$

The result follows. □

By Theorems 2.1 and 2.2, we have

Theorem 2.3. *Let G be a connected graph with n vertices and m edges. Then*

$$r_L(G)\geq\frac{1}{4}\left(\frac{\sqrt{(n-1)(M_1+2m)}}{m}+\frac{\sqrt{(n-1)(M_1+2m)-4m^2}}{m}\right)^2.$$

By Theorem 2.3, it is easy to obtain the following.

Corollary 2.2. *Let G be a connected k-regular graph with n vertices. Then*

$$r_L(G)\geq\left(\sqrt{\frac{(n-1)(k+1)}{nk}}+\sqrt{\frac{(n-1)(k+1)}{nk}-1}\right)^2.$$

For a connected k-regular ($k\leq n-2$) graph with n vertices, by Theorem A, then

$$r_L(G)\geq\frac{k+1}{k}. \tag{2}$$

Remark 2.2. Let

$$f(k) := \left(\sqrt{\frac{(n-1)(k+1)}{nk}} + \sqrt{\frac{(n-1)(k+1)}{nk} - 1}\right)^2 - \frac{k+1}{k}$$
$$= \frac{n-2k-2}{nk} + 2\sqrt{\frac{(n-1)(k+1)}{nk}}\sqrt{\frac{(n-1)(k+1)}{nk} - 1}.$$

We only need to show $(2\sqrt{\frac{(n-1)(k+1)}{nk}}\sqrt{\frac{(n-1)(k+1)}{nk}-1})^2 \geq (\frac{2k+2-n}{nk})^2$. The inequality can be transformed into $n(4k(n-2-k)+3n-4) \geq 0$. This inequality is evidently satisfied for $k \leq n-2$. Then $f(k) > 0$, i.e., the bound of Corollary 2.2 is better than (2).

Remark 2.3. By Appendix A of [7], for a tree T with n $(3 \leq n \leq 10)$ vertices, then $r_L(S_n) = n \leq r_L(T) \leq r_L(P_n)$.

By Remark 2.3, it is naturally to conjecture that:

Conjecture 2.1. *Let T be a tree with n $(n \geq 3)$ vertices, then $r_L(S_n) = n \leq r_L(T) \leq r_L(P_n)$ and the left (right) equality holds if and only if $G \cong S_n (G \cong P_n)$.*

Lemma 2.3 ([8]). *If T is a tree with diameter $D(T)$, then $\alpha(T) \leq 2(1-\cos(\frac{\pi}{D(T)+1}))$.*

Lemma 2.4 ([8]). *Let $T \neq K_{1,n-1}$ be a tree on $n \geq 6$ vertices, then $\alpha(G) = \mu_{n-1}(T) < 0.49$.*

Lemma 2.5 ([9]). *Let G be a graph with at least one edge and maximum vertex degree Δ. Then $\mu_1 \geq \Delta + 1$ with equality for connected graph if and only if $\Delta = n-1$.*

Proposition 2.1. *Let T be a tree with n $(n \geq 10)$ vertices. If $\Delta(T) \geq \lceil \frac{n}{2} \rceil - 1$ or $D(T) \geq \lceil \frac{n}{2} \rceil - 1$, where $D(T)$ is the diameter of T, then $r_L(T) > n = r_L(S_n)$.*

Proof. If $\Delta(T) \geq \lceil \frac{n}{2} \rceil - 1$, by Lemmas 2.4 and 2.5, then $r_L(T) \geq \frac{\Delta+1}{\mu_{n-1}} > \frac{\lceil \frac{n}{2} \rceil - 1 + 1}{0.49} > n$.

If $D(T) \geq \lceil \frac{n}{2} \rceil - 1$, note that $\mu_1(T) \geq \mu_1(P_n) = 2(1+\cos\frac{\pi}{n})$ and by Lemma 2.3, then

$$r_L(T) \geq \frac{2\left(1+\cos\frac{\pi}{n}\right)}{2\left(1-\cos\left(\frac{\pi}{D(T)+1}\right)\right)} \geq \frac{1+\cos\frac{\pi}{n}}{1-\cos\frac{2\pi}{n}} = \frac{1}{2\left(1-\cos\frac{\pi}{n}\right)}.$$

Let $f(x) := x(1-\cos\frac{\pi}{x}) - \frac{1}{2}$.

Consider the first derivative $f'(x) = 1 - \cos\frac{\pi}{x} - \frac{\pi}{x}\sin\frac{\pi}{x} = 2\sin\frac{\pi}{2x}\cos\frac{\pi}{2x}(\tan\frac{\pi}{2x} - \frac{\pi}{x})$.

Let $g(y) := \tan y - 2y$. And $g'(y) = \frac{(\sin y - \cos y)(\sin y + \cos y)}{\cos^2 y} < 0$ for $y < \frac{\pi}{4}$. Then $g(y)$ is decreasing function on y. Thus $g(\frac{\pi}{2x}) = \tan\frac{\pi}{2x} - \frac{\pi}{x} < g(0) = 0$ for $x > 0$.

Then $f(x)$ is a decreasing function on x and $f(x) \leq f(10) \doteq -0.0105 < 0$, i.e., $\frac{1}{2(1-\cos\frac{\pi}{x})} > x$ for $x \geq 10$. Hence $r_L(T) \geq \frac{1}{2(1-\cos\frac{\pi}{n})} > n$.

The result follows. □

Lemma 2.6 ([10]). *Let G be a graph with n vertices. Then $\mu_1(G) = \mu_2(G) = \cdots = \mu_{n-1}(G)$ if and only if $G \cong K_n$ or $G \cong \overline{K_n}$.*

Theorem 2.4. *Let G be a connected graph with n $(n \geq 3)$ vertices. Then $r_L(G) \geq 1$ with the equality holds if and only if $G \cong K_n$.*

Proof. Since $\mu_1(G) \geq \alpha(G)$, $r_L(G) = \frac{\mu_1(G)}{\alpha(G)} \geq 1$. The equality holds if and only if $\mu_1(G) = \alpha(G)$, i.e., $\mu_1(G) = \mu_2(G) = \cdots = \mu_{n-1}(G)$.

By Lemma 2.6, the result holds. □

The following property [11] of the Laplacian eigenvalues is needed.

Let $\overline{G}$ (or G^c) be the complement of the graph G with n vertices. The Laplacian eigenvalues of $\overline{G}$ are $n-\mu_{n-1}, n-\mu_{n-2}, \ldots, n-\mu_1, 0$.

Theorem 2.5. *Let $G \neq K_n$ be a connected graph with n vertices. Then $r_L(G) \geq \frac{n}{n-2}$. The equality holds if and only if $G \in \mathcal{G}$, where $\mathcal{G}$ is the set of graphs such that $\overline{G} = iK_2 \cup (n-2i)K_1$ $(i = 1, \ldots, \lfloor \frac{n}{2} \rfloor)$.*

Proof. There are two cases:

Case 1. $\Delta - \delta \geq 1$.

By Theorem A, $r_L(G) \geq \frac{\Delta+1}{\delta} \geq \frac{\delta+2}{\delta} \geq \frac{n}{n-2}$. The last two equality hold if and only if $\Delta = \delta + 1$ and $\delta = n - 2$, i.e., $\Delta = n - 1$ and $\delta = n - 2$.

Then $\Delta(\overline{G}) = 1$ and $\delta(\overline{G}) = 0$. Hence $\overline{G} = iK_2 \cup (n - 2i)K_1$.

Case 2. $\Delta = \delta$.

Then G is a k-regular graph. If $k < n - 2$, similar to Remark 2.1 and by Corollary 2.2, then we have $r_L \geq (\sqrt{\frac{(n-1)(k+1)}{nk}} + \sqrt{\frac{(n-1)(k+1)}{nk}} - 1)^2 > \frac{k+2}{k} > \frac{n}{n-2}$.

For a $n - 2$-regular graph G, $\overline{G}$ is the 1-regular graph. Then $\overline{G} \cong \frac{n}{2}K_2$ and n is even. Hence the Laplacian eigenvalues are n ($\frac{n}{2} - 1$ times), $n - 2$ ($\frac{n}{2}$ times) and 0. And $r_L(G) = \frac{n}{n-2}$.

The proof is completed. □

Proposition 2.2. *There exists a connected graph G with n vertices, such that $r_L(G) = \frac{n}{t}$, where t is a positive integer $(1 \leq t \leq n)$ and $t \neq n - 1$.*

Proof. Let $\overline{G} = K_{n-t} \cup tK_1$. Then $\overline{G}$ has the Laplacian eigenvalues: $n - t$ ($n - t - 1$ times) and 0 ($t + 1$ times). And G has the Laplacian eigenvalues: n (t times), t ($n - t - 1$ times) and 0.

Hence $r_L(G) = \frac{n}{t}$. □

Corollary 2.3. *There exist non-isomorphic 2-tough graphs with n $(n \geq 6)$ vertices.*

Proof. In Proposition 2.2, let t_i be positive integers such that $t_i \geq \frac{2n}{3}$, then

$r_L(G) = \frac{n}{t_i} \leq \frac{n}{\frac{2n}{3}} = \frac{3}{2}$. By Theorem B, the proof is completed. □

3. The Laplacian spectral ratio and graph operations

Lemma 3.1 (*[12]*). *For $e \notin E(G)$, the Laplacian eigenvalues of G and $G' = G + e$ interlace, i.e., $\mu_1(G') \geq \mu_1(G) \geq \mu_2(G') \geq \mu_2(G) \geq \cdots \geq \mu_n(G') = \mu_n(G) = 0$.*

Lemma 3.2 (*[13]*). *Let G be a connected graph on n vertices. If v is a pendant vertex of G, then $\mu_i(G) \leq \mu_{i-1}(G - v)$, $2 \leq i \leq n$. Particularly, $\alpha(G) \leq \alpha(G - v)$.*

Theorem 3.1. *Let G be a connected graph with n vertices. If v is a pendant vertex of G, then $r_L(G) \geq r_L(G - v)$.*

Proof. By Lemmas 3.1 and 3.2, $\mu_1(G) \geq \mu_1(G - v)$ and $\alpha(G) \leq \alpha(G - v)$.

Then $r_L(G) = \frac{\mu_1(G)}{\alpha(G)} \geq \frac{\mu_1(G-v)}{\alpha(G-v)} = r_L(G - v)$. □

Remark 3.1. Theorem 3.1 shows that $r_L(G)$ decreases when pendant vertices are deleted. But if we delete a non-pendant vertex, then r_L maybe increase or decrease. For instance, let U_1 be the unicyclic graph obtained from C_4 by attaching a pendent vertex to C_4. S_4 and P_4 are two subgraphs by deleting different vertices. Then $r_L(U_1) \doteq 5.3996$, $r_L(S_4) = 4$ and $r_L(P_4) \doteq 5.82838$.

Lemma 3.3 (*[14]*). *For a connected graph G, $\mu_1(G) = 2\Delta(G)$ if and only if G is a bipartite $\Delta(G)$-regular graph.*

Lemma 3.4 (*[15]*). *Let G be a d-regular simple graph with m edges and n vertices. Then the line graph $L(G)$ has the Laplacian spectrum: $2d$ ($m - n$ times), $\mu_1, \ldots, \mu_{n-1}, \mu_n = 0$.*

Theorem 3.2. *Let G be a connected d-regular graph with n vertices and $L(G)$ the line graph of G. Then $r_L(L(G)) \geq r_L(G)$, the equality holds if and only if G is bipartite d-regular graph.*

Proof. By Lemma 3.4, $r_L(L(G)) = \frac{2d}{\alpha(G)} \geq \frac{\mu_1(G)}{\alpha(G)} = r_L(G)$. The equality holds if and only if $\mu_1 = 2d = 2\Delta$, by Lemma 3.3, i.e., G is a bipartite d-regular graph. □

Theorem 3.3. *Let G be a connected graph with n vertices and $\mu_1 \neq n$. Then*

$r_L(G) \geq r_L(\overline{G})$ if and only if $\mu_1 + \mu_{n-1} \leq n$, i.e., $\mu_1(G) \leq \mu_1(\overline{G})$.

Proof. Since $\mu_1 \neq n$, $\mu_{n-1}(\overline{G}) = n - \mu_1 > 0$.

Then $r_L(G) - r_L(\overline{G}) = \frac{\mu_1}{\mu_{n-1}} - \frac{n-\mu_{n-1}}{n-\mu_1} = \frac{(\mu_1-\mu_{n-1})[n-(\mu_1+\mu_{n-1})]}{\mu_{n-1}(n-\mu_1)}$.

The result follows. □

Corollary 3.1. *Let $T \neq K_{1,n-1}$ be a tree with n $(n \geq 6)$ vertices. Then $r_L(T) > r_L(\overline{T})$.*

Proof. By [16], $\mu_1(T) - \mu_{n-1}(T) \leq n - 1$. By Lemma 2.4,

$$\mu_1(T) + \mu_{n-1}(T) = \mu_1(T) - \mu_{n-1}(T) + 2\mu_{n-1}(T) \leq n - 1 + 2\mu_{n-1}(T) < n - 0.02.$$

Hence the result follows by Theorem 3.3. □

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References

[1] M. Fiedler, Algebraic connectivity of graphs, Czechoslovak Math. J. 23 (1973) 298–305.
[2] M. Barahona, L.M. Pecora, Synchronization in small-world networks, Phys. Rev. Lett. 89 (2002) 054101.
[3] F. Goldberg, Bounding the gap between extremal Laplacian eigenvalues of graphs, Linear Algebra Appl. 416 (2006) 68–74.
[4] B. Liu, S. Chen, Algebraic conditions for t-tough graphs, Czechoslovak Math. J. 60 (135) (2010) 1079–1089.
[5] W.H. Haemers, Interlacing eigenvalues and graphs, Linear Algebra Appl. 227–228 (1995) 593–616.
[6] S.S. Dragomir, A survey on Cauchy–Buniakowsky–Schwartz type discrete inequalities, J. Inequal. Pure Appl. Math. 4 (3) (2003) Article 63.
[7] M.W. Newman, The Laplacian spectrum of graphs, Ph.D. Thesis, Department of Mathematics, University of Manitoba, July 2000.
[8] R. Grone, R. Merris, V.S. Sunder, The Laplacian spectral of graphs, SIAM J. Matrix Anal. Appl. 11 (1990) 218–239.
[9] R. Grone, R. Merris, The Laplacian spectrum of a graph II, SIAM J. Discrete Math. 7 (1994) 221–229.
[10] B. Zhou, On the sum of powers of the Laplacian eigenvalues of graphs, Linear Algebra Appl. 429 (2008) 2239–2246.
[11] R. Merris, Laplacian matrices of graphs: a survey, Linear Algebra Appl. 197–198 (1994) 143–176.
[12] D. Cvetković, M. Doob, H. Sachs, Spectra of Graphs, Academic Press, New York, 1980.
[13] Y. Liu, Y. Liu, The ordering of unicyclic graphs with the smallest algebraic connectivity, Discrete Math. 309 (2009) 4315–4325.
[14] C.D. Godsil, Algebraic Combinatorics, Chapmann and Hall, New York, 1993.
[15] A.K. Kel'mans, Properties of the characteristic polynomial of a graph, in: Kibernetiky-na Službu Kommunizmu, vol. 4, Energija, Moskva-Leningrad, 1967, pp. 27–41 (in Russian).
[16] Y. Fan, J. Xu, Y. Wang, D. Liang, The Laplacian spread of a tree, Discrete Math. Theoret. Comput. Sci. 10 (2008) 79–86.